Crossover properties from random percolation to frustrated percolation

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We investigate the crossover properties of the frustrated percolation model on a two-dimensional square lattice, with asymmetric distribution of ferromagnetic and antiferromagnetic interactions. We determine the critical exponents ν, γ, and β of the percolation transition of the model, for various values of the density of antiferromagnetic interactions π in the range $0 \leq \pi \leq 0.5$. Our data are consistent with the existence of a crossover from random percolation behavior for $\pi = 0$, to frustrated percolation behavior, characterized by the critical exponents of the ferromagnetic 1/2-state Potts model, as soon as $\pi > 0$.

I. INTRODUCTION

The cluster approach was introduced by Kasteleyn and Fortuin (KF) [1] and Coniglio and Klein (CK) [2] in ferromagnetic spin systems. In the Coniglio-Klein approach, one puts a bond between two nearest neighbor (NN) spins with probability $p = 1 - e^{-2\beta J}$ if the spins are parallel, where J is the spin interaction and $\beta = 1/k_B T$, and with probability zero if they are antiparallel. In this way a bond configuration C on the lattice has a statistical weight

$$W(C) = e^{\mu b(C)} q^{N(C)}, \qquad (1)$$

where $\mu = \log\left(\frac{p}{1-p}\right) = \log(e^{2\beta J} - 1)$ is the chemical potential of the bonds, $b(C)$ is the number of bonds and $N(C)$ the number of clusters of the configuration C, and q is the multiplicity of the spins ($q = 2$ for Ising spins). This defines a percolation model, in which clusters percolate at the ferromagnetic critical point, with the same critical indices of the original spin model.

The KF-CK approach has been extended also to frustrated spin models, as for example spin glasses [3]. In these models the disorder is produced by a quenched distribution of antiferromagnetic $(-J)$ and ferromagnetic (J) interactions on the lattice. Just like in the ferromagnetic case, one puts a bond between two NN spins if they satisfy the interaction, with a probability $p = 1 - e^{-2\beta J}$. The main difference here is that, due to frustration, the spins cannot satisfy simultaneously all the interactions on the lattice. More specifically, if a closed path on the lattice contains an odd number of antiferromagnetic interactions, not all the interactions belonging to the path can be satisfied simultaneously. Such a path is called "frustrated loop", and since bonds can be put only between spins that satisfy the interactions, a frustrated loop cannot be completely occupied by bonds. Therefore, the statistical weight of a bond configuration C will be now

$$W(C) = \begin{cases} e^{\mu b(C)} q^{N(C)} & \text{if } C \text{ is not frustrated,} \\ 0 & \text{if } C \text{ is frustrated,} \end{cases} \qquad (2)$$

where C is said frustrated if it contains one or more frustrated loops completely occupied by bonds.

It has been shown by renormalization group methods on a hierarchical lattice [4] that this model exhibits two phase transitions, for every value of the multiplicity q of the spins. The first transition, at a temperature $T_{SG}(q)$, is in the universality class of the Ising SG transition, while the other transition, at a temperature $T_p(q) > T_{SG}(q)$, is a percolation transition in the universality class of the ferromagnetic $q/2$-state Potts model [5].

The frustrated percolation model has proven to be a suitable model to the study of complex systems, such as spin glasses [6], glasses [7] and in general all those systems in which connectivity and frustration play a fundamental role, (for a review see [8]).

The aim of the present paper is the study of the percolation transition of the model with $q = 1$, the "bond frustrated percolation model", for a variable density π of antiferromagnetic interactions in the interval $0 \leq \pi \leq 0.5$. We determine the critical probability $p_c(\pi)$ and the critical exponents $\nu(\pi)$, $\beta(\pi)$, and $\gamma(\pi)$, by performing Monte Carlo simulations on lattices of different size, and using scaling laws (see Sect. III). Finally the results are compared with the theoretical predictions for the two extreme cases, the pure ferromagnetic case $\pi = 0$, that corresponds to random bond percolation ($1/\nu = 0.75$, $\beta/\nu = 0.1042$, $\gamma/\nu = 1.7917$), and the symmetric case $\pi = 0.5$, that corresponds to the 1/2-state Potts model ($1/\nu = 0.5611$, $\beta/\nu = 0.08276$, $\gamma/\nu = 1.8346$).

II. DEFINITION OF THE FRUSTRATED PERCOLATION MODEL

The bond frustrated percolation model can be defined in the following way. Consider a two-dimensional square lattice, with NN interactions between sites. These interactions can be ferromagnetic with probability $1 - \pi$ and antiferromagnetic with probability π. The distribution of interactions is quenched, so it is set at the beginning and does not evolve with time in the dynamics of the system.

Each edge of the lattice, connecting a pair of NN sites, can be connected by a bond or not, and the state of the system is completely specified by the bond configuration.

We give to a bond configuration C a statistical weight

$$W(C) = \begin{cases} e^{\mu b(C)} & \text{if } C \text{ is not frustrated,} \\ 0 & \text{if } C \text{ is frustrated,} \end{cases} \quad (3)$$

where $\mu = \log\left(\frac{p}{1-p}\right)$, p is a probability that is connected to the temperature via the relation $p = 1 - e^{-2\beta J}$, and $b(C)$ is the number of bonds of the configuration C.

The presence of frustration induces a complex behavior in both the static and dynamic properties of the system, with the presence of many metastable states, and high free energy barriers separating them. For $\pi = 0$ the model coincides with random percolation. For $\pi = 1/2$, that is equal density of ferromagnetic and antiferromagnetic interactions, the system is expected to have two phase transitions [4]. The first at lower temperature in the same universality class of the Ising spin glass transition, which in two dimensions is at $T = 0$, that is at $p = 1$. Below this temperature the system is frozen, ergodicity is broken and the system remains trapped in a finite region of phase space. The second transition at a higher temperature (lower probability), is the percolation transition of the cluster of bonds, and belongs to a different universality class, namely that of the ferromagnetic 1/2-state Potts model.

We will study the percolation transition as a function of the density of antiferromagnetic interactions π. We will see that a very small amount of antiferromagnetic interactions is already sufficient to change the universality class of the transition.

III. MONTE CARLO RESULTS

We have studied the percolation transition of the model on a 2D square lattice with periodic boundary conditions, for different values of π ($0 \le \pi \le 0.5$) and for different lattice sizes L ($L = 16, 24, 32, 40$, occasionally larger). For each size of the lattice a configuration of interactions is preliminary produced by setting randomly on the lattice a number N_a of antiferromagnetic interactions and $N_f = 2L^2 - N_a$ of ferromagnetic interactions. The corresponding density of antiferromagnetic interactions is given by $\pi = N_a/2L^2$.

We have determined the critical probability of percolation $p_c(\pi)$ and the critical exponents $\nu(\pi)$, $\gamma(\pi)$, and $\beta(\pi)$, by performing Monte Carlo simulations and using scaling laws. In order to generate bonds configurations with the appropriate weights we used an algorithm devised by Sweeny [9], suitably modified to treat the frustration occurrence [10]. For each π the quantities of interest were averaged on many configurations of interactions.

For each value of π the critical probability of percolation has been determined using the following scaling law for the probability P_∞ of a spanning cluster being present on the lattice [11]

$$P_\infty(L, p) = \widetilde{P}_\infty[L^{1/\nu}(p - p_c)], \quad \text{for } L \to \infty, p \to p_c. \quad (4)$$

Therefore p_c is found to be the point where the curves of P_∞ as a function of p, with different lattice sizes, intersect.

In Fig. 1 are shown the curves $P_\infty(L, p)$ for $\pi = 0.5$, and in Fig. 2 the values obtained for $p_c(\pi)$. The intersection point is clearly singled out. This procedure enabled us to determine p_c with an indetermination of ± 0.001, or in the worst cases of ± 0.002.

The simulations were performed on lattices of size $L = 16, 24, 32, 40$, for each L about 500 MC steps were produced to thermalize the system whereas $20,000$-$30,000$ MC steps were used to average. Moreover for each value of π we have averaged on a number n of configurations of interactions ranging from $n = 80$ for $L = 16$ to $n = 30$ for $L = 40$.

The scaling law in Eq. (4) enables us to get the value of the exponent $1/\nu$ as well, once p_c has been determined, by choosing the value which gives the best data collapse of the curves (see Fig. 3). In Fig. 4 we give the values of $1/\nu(\pi)$ obtained. As π increases from zero to positive values, a sudden change in the universality class of the transition can be seen. A crossover region extending from $\pi = 0$ to $\pi \simeq 0.05$ occurs. From this point the value of ν can be considered in agreement with the predicted value for the 1/2-state Potts model ($1/\nu = 0.56$, marked by the horizontal straight line).

The errors $\Delta(1/\nu)$ and Δp_c were computed as the amplitudes of the intervals in the values of $1/\nu$ and p_c for which a good data collapse was obtained. We remark that these errors do not take into account the finiteness of the system. Thus the values for the critical exponents must be regarded as effective values which would give correct results only in the asymptotic limit ($L \to \infty$).

The mean cluster size χ is defined as

$$\chi = \frac{1}{V} \sum s^2 n_s, \quad (5)$$

where $V = L^2$ is the number of sites on the lattice, n_s is the number of clusters of size s, and the sum extends over the cluster sizes s. In the thermodynamic limit $L \to \infty$ the mean cluster size diverges as $|p - p_c|^{-\gamma}$, when the probability p approaches its critical value. For finite systems, χ obeys a finite size scaling [11]

$$\chi(L, p) = L^{\gamma/\nu} \widetilde{\chi}[L^{1/\nu}(p - p_c)] \quad \text{for } L \to \infty, p \to p_c. \quad (6)$$

The density of the largest cluster ρ_∞ plays the role of the order parameter in the system, being zero for $p < p_c$ in the thermodynamic limit, and $\rho_\infty \propto (p - p_c)^\beta$ for $p > p_c$. This quantity as well obeys a finite size scaling

$$\rho_\infty(L, p) = L^{-\beta/\nu} \widetilde{\rho}_\infty[L^{1/\nu}(p - p_c)] \quad \text{for } L \to \infty, p \to p_c. \quad (7)$$

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Therefore simulating the system at the computed value of p_c enables us to get γ and β from a log-log plot of the relations $\chi(L, p = p_c) \propto L^{\gamma/\nu}$, $\rho_\infty(L, p = p_c) \propto L^{-\beta/\nu}$. In Fig. 5 one such plot is shown, for $\pi = 0$, while in Fig. 6 and Fig. 7 are shown the plots of γ/ν and β/ν obtained, for different values of π. The horizontal straight lines mark the predicted values of the exponents, $\gamma/\nu = 1.83$ and $\beta/\nu = 0.083$, corresponding to the ferromagnetic 1/2-state Potts model. We see that for $\pi > 0.05$ the computed value of the exponents is in good agreement with the prediction.

These results are consistent with the picture that there are two universality classes: random percolation at $\pi = 0$, and frustrated percolation for $\pi > 0$. Data at our disposal cannot exclude however the possibility that the random percolation behavior extends from $\pi = 0$ to a value π^* smaller than 0.05.

IV. CONCLUSIONS

We have investigated the percolation transition of the asymmetric frustrated percolation model in two dimensions by using Monte Carlo simulation.

From the analysis of critical exponents $\nu(\pi)$, $\gamma(\pi)$, $\beta(\pi)$, in the interval $0 \leq \pi \leq 0.5$, it seems reasonable to assume that a very small concentration ($\pi \simeq 0.05$) of antiferromagnetic interactions is already sufficient to produce the change in the universality class of the transition. This means that the effects of disorder and frustration are important even for such small values of π.

Our results are consistent with the existence of a sharp crossover from random percolation for $\pi = 0$, to frustrated percolation, characterized by the exponents of the ferromagnetic 1/2-state Potts model, as soon as $\pi > 0$. However we cannot rule out numerically the presence of a tricritical point, at low values of π, dividing random percolation from frustrated percolation exponents.

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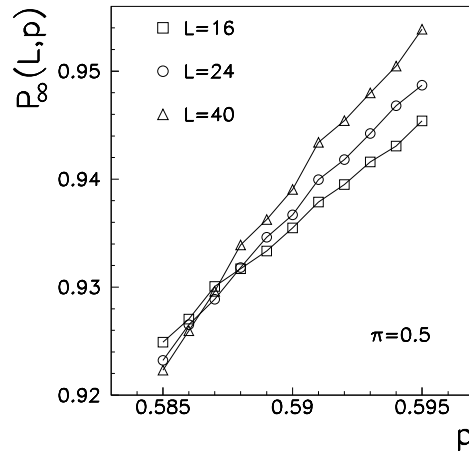[11] A. Aharony, and D. Stauffer, *Introduction to Percolation Theory*, Taylor & Francis (1994).

FIG. 1. Probability of percolation P_∞, as a function of the probability p, for lattice sizes $L = 16, 24, 40$ and for $\pi = 0.5$. The critical probability p_c is found to be the intersection point of the different curves.
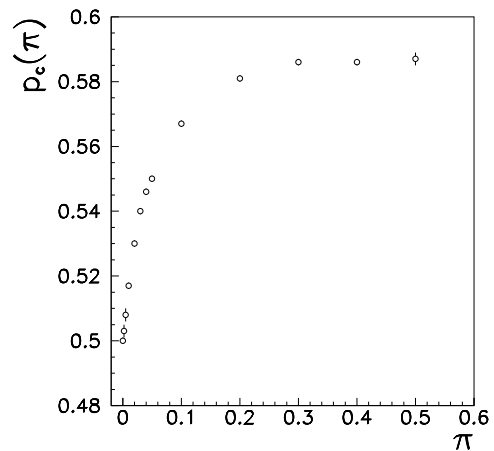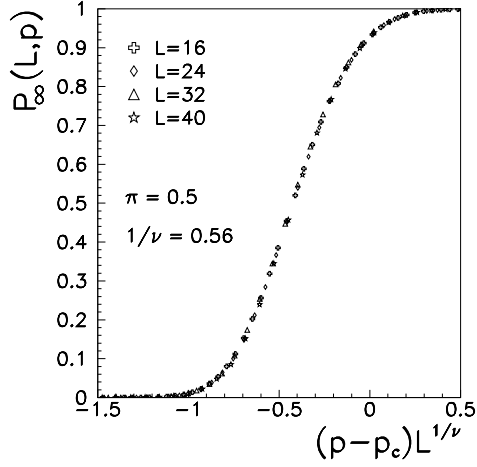

FIG. 2. The critical probability of percolation p_c as a function of the density of antiferromagnetic interactions π. The value $p_c = 0.5$ found for $\pi = 0$ is in agreement with the value of the random percolation.

FIG. 3. Scaling behavior of the probability of percolation P_∞, for $\pi = 0.5$. The exponent $1/\nu$ is found as the value which gives the best data collapse. The value $1/\nu = 0.56$ found is in agreement with the value of the 1/2-state Potts model.
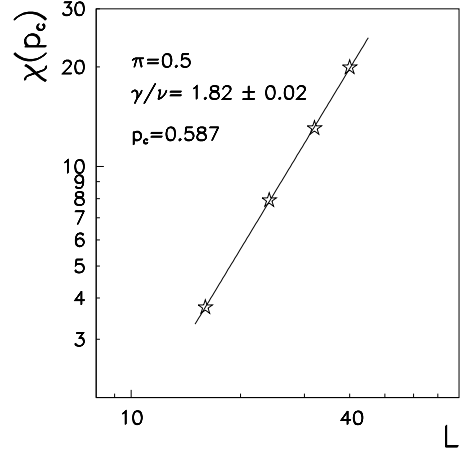

FIG. 5. Log-log plot of the mean cluster size χ in function of L, at p_c and for $\pi = 0.5$, and linear fit of the data. The critical exponent γ/ν is found to be the slope of the straight line.
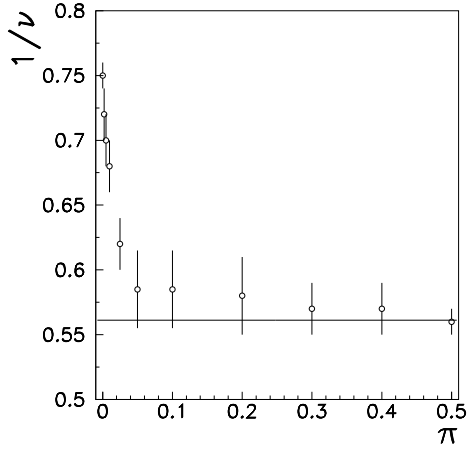

FIG. 4. Plot of the exponent $1/\nu$ in function of the density of antiferromagnetic interactions π. The horizontal straight line represents the value of $1/\nu$ for the frustrated percolation.
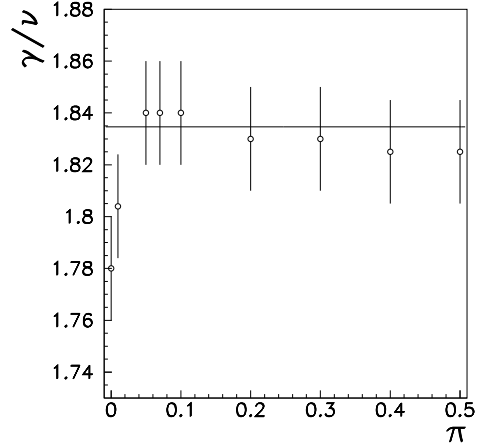

FIG. 6. Plot of the exponent γ/ν in function of the density of antiferromagnetic interactions π. The horizontal straight line represents the value of γ/ν for the frustrated percolation.
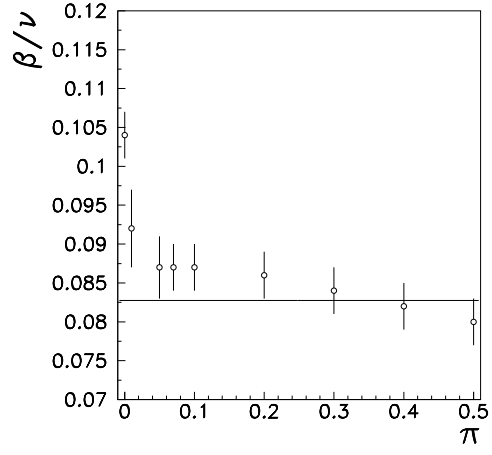
FIG. 7. Plot of the exponent β/ν in function of the density of antiferromagnetic interactions π. The horizontal straight line represents the value of β/ν for the frustrated percolation.